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On Energy of Prime Ideal Graph of a Commutative Ring Associated with Transmission-Based Matrices

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Abstract

This research explores the energy of the prime ideal graph of a commutative ring. The study demonstrates the energy formula of the graph associated with transmission-based matrices. Through research, the findings highlight the distance, Wiener-Hosoya, and distance signless Laplacian matrices. It should be noted that the distance and Wiener-Hosoya energies are always twice their spectral radius, meanwhile, it does not hold for distance signless Laplacian energy.

Keywords: prime ideal graph; energy of a graph; commutative ring; distance matrix; distance signless Laplacian matrix; Wiener-Hosoya matrix.

1 Introduction

Ring theory originated in abstract algebra in the early 19th century when commutative and non-commutative rings were being investigated. In mathematics, rings are fundamental structures composed of sets with two binary operations, addition, and multiplication [2]. Graphs from rings have been interesting to study in the last 30 years. One of these graphs is the prime ideal graph.

There are several graphs whose vertex set is a group or a ring. Anderson et al. [1] wrote a book about graphs associated with commutative rings. Several results on graphs defined on rings or groups can be found in several papers. Zai et al. [18] focus on finite commutative rings for non-zero divisor graphs, meanwhile, the prime graph discussion can be seen in [7]. Apart from the ring, Romdhini et al. [12] investigated the dihedral groups as the vertex set of commuting and non-commuting graphs. The association between graph and lattice is presented by Malekpour and Bazigaran [8]. Romdhini et al. [14] explored the spectral properties of the power graph of dihedral groups. Rehman et al. [11] also investigated the eigenvalues of the zero-divisor graph of the ring based on the normalized distance Laplacian matrix. The connection between the zero divisor and prime graphs was observed in [8]. In 2022, the prime ideal graph definition was first introduced by Salih and Jund [16] as the following definition.

Definition 1.1. [16] *The prime ideal graph is denoted by $\Gamma(R, P)$ where R is any commutative ring and P is its prime ideal. The vertex set is $R\backslash\{0\}$ and two distinct vertices u and v are adjacent whenever $uv \in P$.*

The graph energy concept was first defined by Gutman [3] in 1978. It is defined as the sum of absolute eigenvalues of a graph. This definition is based on the adjacency matrix of a graph. This paper devotes the transmission-based matrices of a graph including the Wiener-Hosoya and distance signless Laplacian matrices. In 2021, Ibrahim et al. [5] pioneered the Wiener-Hosoya matrix definition of a graph. Later, Pirzada and Haq [9] defined the distance signless Laplacian matrix of a graph, which involves the distance and transmission matrices. Then, Romdhini et al. [15] extend this study to formulate the Wiener-Hosoya energy of the non-commuting graph for dihedral groups. Several distance-based matrices have been applied in [13] which discuss the degree sum exponent distance energies.

Throughout this work, we correspond $\Gamma(R, P)$ with distance, Wiener-Hosoya, and distance signless Laplacian matrices. The primary goal is spectral radius and energy formulations and analyzing their relationship.

2 Preliminaries

The basic concepts and definitions are briefly described in this section. Let $|R| = n$ and $|P| = m$. There are $n-1$ vertices in $\Gamma(R, P)$. Let d_{pq} be the distance between vertex v_p and v_q, and d_p be the degree of vertex v_p. The following result presents the degree formula of $v_p \in \Gamma(R, P)$.

Theorem 2.1. [17] *The degree of vertex v_p in $\Gamma(R, P)$ is*

$$d_p = \begin{cases} n-2, & \text{for every } v_p \in P\backslash\{0\}, \\ m-1, & \text{for every } v_p \in R\backslash P. \end{cases}$$

Afterward, the distance between two vertices was explored in [17].

Theorem 2.2. [17] *The distance between two vertices v_p and v_q in $\Gamma(R,P)$ is given by*

$$d_{pq} = \begin{cases} 1, & \text{for every } v_p \in P\backslash\{0\} \text{ and } v_q \in R, \\ 2, & \text{for every } v_p, v_q \in R\backslash P. \end{cases}$$

Furthermore, for $v_p \in \Gamma(R,P)$, let τ_p be the transmission of v_p, which is defined as the sum of d_{pq}, for all $v_q \in \Gamma(R,P)$ [5].

Definition 2.1. [5] *The Wiener-Hosoya (WH) matrix corresponding to $\Gamma(R,P)$ is written by $WH\left(\Gamma(R,P)\right) = [wh_{pq}]$ with entries are*

$$wh_{pq} = \begin{cases} \frac{\tau_p}{2d_p} + \frac{\tau_q}{2d_q}, & \text{if } v_p \neq v_q \text{ and they are adjacent}, \\ 0, & \text{otherwise}. \end{cases}$$

Definition 2.2. [6] *The distance matrix of $\Gamma(R,P)$, $D(\Gamma(R,P)$, is a square matrix whose entries are d_{pq} for $v_p \neq v_q$, and zero if $v_p = v_q$.*

Definition 2.3. [9] *The distance signless Laplacian (DSL) matrix of $\Gamma(R,P)$ is given by*

$$DSL\left(\Gamma(R,P)\right) = T\left(\Gamma(R,P)\right) + D\left(\Gamma(R,P)\right),$$

where $T\left(\Gamma(R,P)\right) = diag\left(\tau_{v_1}, \tau_{v_2}, \ldots, \tau_{v_n}\right)$.

The spectrum of $\Gamma(R,P)$ corresponding to the Wiener-Hosoya matrix can be written as:

$$Spec_{WH}(\Gamma(R,P)) = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ k_1 & k_2 & \ldots & k_n \end{bmatrix},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $WH(\Gamma(R,P))$ and $k_1, k_2, \ldots, k_n$ are their respective multiplicities. Therefore, the Wiener-Hosoya energy of $\Gamma(R,P)$ [3] can be defined as follows:

$$E_{WH}(\Gamma(R,P)) = \Sigma_{i=1}^{n} \left|\lambda_i\right|,$$

and spectral radius of $\Gamma(R,P)$ [4] associated with the adjacency matrix is defined as

$$\rho_{WH}(\Gamma(R,P)) = max\left\{|\lambda| : \lambda \in Spec_{WH}(\Gamma(R,P))\right\}.$$

The above notations also apply to the distance and DSL-matrices. Furthermore, we require the following result to formulate the characteristic polynomial of $\Gamma(R,P)$.

Lemma 2.1. [10] *If a, b, c, and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\lambda+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda+b)I_{n_2} - bJ_{n_2} \end{vmatrix},$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$$(\lambda+a)^{n_1-1}(\lambda+b)^{n_2-1}\left((\lambda-(n_1-1)a)(\lambda-(n_2-1)b) - n_1n_2cd\right).$$

3 Main Results

To construct the Wiener-Hosoya and distance signless Laplacian matrices of $\Gamma(R, P)$, according to Definition 2.1 and 2.3, we need the vertex transmission of $\Gamma(R, P)$.

Theorem 3.1. *The transmission of v_p in $\Gamma(R, P)$ is*

$$\tau_p = \begin{cases} n-2, & \text{for every } v_p \in P\backslash\{0\}, \\ 2n-m-3, & \text{for every } v_p \in R\backslash P. \end{cases}$$

Proof. Let $R\backslash\{0\} = \{p_1, p_2, \ldots, p_m, r_1, r_2, \ldots, r_{n-m-1}\}$ and $P = \{p_1, p_2, \ldots, p_m\}$. Based on Theorem 2.2 and the connectivity of $\Gamma(R, P)$, then we have two cases. The first case when $v_p \in P\backslash\{0\}$ and $v_q \in P\backslash\{0\}$, the total distances from vertex v_p to v_q is $m-2$, and when $v_q \in R\backslash P$, the total distance is $n-m$. Hence,

$$\tau_p = m - 2 + n - m = n - 2.$$

Meanwhile, for the second case when $v_p \in R\backslash P$ and $v_q \in P\backslash\{0\}$, the total distances between vertex v_p and v_q is $m-1$, and if $v_q \in R\backslash P$, the total is $2(n-m-1)$. Therefore,

$$\tau_p = m - 1 + 2(n - m - 1) = 2n - m - 3.$$

□

3.1 Distance energy

In this part, we demonstrate the distance energy of $\Gamma(R, P)$.

Theorem 3.2. *The characteristic polynomial of $D(\Gamma(R, P))$ is*

$$P_{D(\Gamma(R,P))}(\lambda) = (\lambda + 2)^{n-m-2}(\lambda + 1)^{m-1}\left(\lambda^2 - (2n - m - 5)\lambda + m(n - m - 1) - 2(n - 2)\right).$$

Proof. Let $R\backslash\{0\} = \{p_1, p_2, \ldots, p_m, r_1, r_2, \ldots, r_{n-m-1}\}$ and $P = \{p_1, p_2, \ldots, p_m\}$. We have $n-1$ vertices for $\Gamma(R, P)$. By Definition 2.2 and Theorem 2.2, we obtain the distance matrix of $\Gamma(R, P)$ as $(n-1) \times (n-1)$ matrix as follows:

$$D(\Gamma(R,P)) = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_m \\ r_1 \\ r_2 \\ \vdots \\ r_{n-m-1} \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & \ldots & p_m & r_1 & r_2 & \ldots & r_{n-m-1} \end{matrix} \\ \begin{pmatrix} 0 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & 2 & \ldots & 2 \\ 1 & 1 & \ldots & 1 & 2 & 0 & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 & 2 & 2 & \ldots & 0 \end{pmatrix} \end{array}. \quad (1)$$

Equation (1) can be partitioned into 4 block matrices as follows:

$$D(\Gamma(R,P)) = \begin{pmatrix} (J-I)_m & J_{m\times(n-m-1)} \\ J_{(n-m-1)\times m} & 2(J-I)_{n-m-1} \end{pmatrix}.$$

The characteristic polynomial of $D(\Gamma(R,P))$ is presented below:

$$P_{D(\Gamma(R,P))}(\lambda) = \begin{vmatrix} (\lambda+1)I_m - J_m & -J_{m\times(n-m-1)} \\ -J_{(n-m-1)\times m} & (\lambda+2)I_{n-m-1} - 2J_{n-m-1} \end{vmatrix}.$$

By Lemma 2.1 with $a=1$, $b=2$, $c=d=1$, $n_1=m$, and $n_2=n-m-1$, then we get

$$P_{D(\Gamma(R,P))}(\lambda) = (\lambda+2)^{n-m-2}(\lambda+1)^{m-1}\left(\lambda^2-(2n-m-5)\lambda+m(n-m-1)-2(n-2)\right).$$

□

Theorem 3.3. *The spectral radius of* $\Gamma(R,P)$ *associated with the distance matrix is*

$$\rho_D(\Gamma(R,P)) = \frac{2n-m-5+\sqrt{(2n-m-5)^2-4m(n-m-1)+8(n-2)}}{2}.$$

Proof. Based on Theorem 3.2, the roots of $P_{D(\Gamma(R,P))}(\lambda)=0$ are eigenvalues of $D(\Gamma(R,P))$. Therefore, we obtain $\lambda_1 = -1$ with multiplicity $m-1$, $\lambda_2 = -2$ of multiplicity $n-m-2$, and $\lambda_{3,4} = \frac{2n-m-5\pm\sqrt{(2n-m-5)^2-4m(n-m-1)+8(n-2)}}{2}$. According to this fact, we get the spectrum of $\Gamma(R,P)$ as follows:

$$Spec_D(\Gamma(R,P)) = \begin{bmatrix} \lambda_3 & -1 & -2 & \lambda_4 \\ 1 & m-1 & n-m-2 & 1 \end{bmatrix}.$$

This leads to the spectral radius of $\Gamma(R,P)$ as

$$\rho_D(\Gamma(R,P)) = \frac{2n-m-5+\sqrt{(2n-m-5)^2-4m(n-m-1)+8(n-2)}}{2},$$

and we complete the proof. □

Theorem 3.4. *The distance energy of* $\Gamma(R,P)$ *is*

$$E_D(\Gamma(R,P)) = 2n-m-5+\sqrt{(2n-m-5)^2-4m(n-m-1)+8(n-2)}.$$

Proof. According to the spectrum of $\Gamma(R,P)$ in the proofing part of Theorem 3.3, the distance energy of $\Gamma(R,P)$ can be obtained as

$$\begin{aligned} E_D(\Gamma(R,P)) =& (n-m-2)|-2| + (m-1)|-1| + \\ & \left|\frac{2n-m-5\pm\sqrt{(2n-m-5)^2-4m(n-m-1)+8(n-2)}}{2}\right| \\ =& 2n-m-5+\sqrt{(2n-m-5)^2-4m(n-m-1)+8(n-2)}. \end{aligned}$$

□

3.2 Wiener-Hosoya energy

This section presents the energy of $\Gamma(R,P)$ associated with the Wiener-Hosoya matrix.

Theorem 3.5. *The characteristic polynomial of* $WH(\Gamma(R,P))$ *is*

$$P_{WH(\Gamma(R,P))}(\lambda)=\lambda^{n-m-2}(\lambda+1)^{m-1}\left(\lambda^2-(m-1)\lambda-\frac{m(n-m-1)(n-2)^2}{(m-1)^2}\right).$$

Proof. By the same argument of the proofing part of Theorem 3.2, we have the vertex set of $\Gamma(R,P)$ as $\{p_1,p_2,\ldots,p_m,r_1,r_2,\ldots,r_{n-m-1}\}$. By Definition 2.1, the vertex transmission in Theorem 3.1, and d_{pq} in Theorem 2.2, we obtain the Wiener-Hosoya matrix of $\Gamma(R,P)$ as given below:

$$WH(\Gamma(R,P))=\begin{array}{c} \\ p_1\\ p_2\\ \vdots\\ p_m\\ r_1\\ r_2\\ \vdots\\ r_{n-m-1}\end{array}\begin{array}{c}\begin{array}{cccccccc} p_1 & p_2 & \cdots & p_m & r_1 & r_2 & \cdots & r_{n-m-1}\end{array}\\ \begin{pmatrix} 0 & 1 & \cdots & 1 & \frac{n-2}{m-1} & \frac{n-2}{m-1} & \cdots & \frac{n-2}{m-1}\\ 1 & 0 & \cdots & 1 & \frac{n-2}{m-1} & \frac{n-2}{m-1} & \cdots & \frac{n-2}{m-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 0 & \frac{n-2}{m-1} & \frac{n-2}{m-1} & \cdots & \frac{n-2}{m-1}\\ \frac{n-2}{m-1} & \frac{n-2}{m-1} & \cdots & \frac{n-2}{m-1} & 0 & 0 & \cdots & 0\\ \frac{n-2}{m-1} & \frac{n-2}{m-1} & \cdots & \frac{n-2}{m-1} & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{n-2}{m-1} & \frac{n-2}{m-1} & \cdots & \frac{n-2}{m-1} & 0 & 0 & \cdots & 0\end{pmatrix}\end{array}$$

$$=\begin{pmatrix}(J-I)_m & \frac{n-2}{m-1}J_{m\times(n-m-1)}\\ \frac{n-2}{m-1}J_{(n-m-1)\times m} & 0_{n-m-1}\end{pmatrix}.$$

The characteristic polynomial of $WH(\Gamma(R,P))$ is presented below:

$$P_{WH(\Gamma(R,P))}(\lambda)=\begin{vmatrix}(\lambda+1)I_m-J_m & -\frac{n-2}{m-1}J_{m\times(n-m-1)}\\ -\frac{n-2}{m-1}J_{(n-m-1)\times m} & \lambda I_{n-m-1}\end{vmatrix}.$$

By Lemma 2.1 with $a=1$, $b=0$, $c=d=\frac{n-2}{m-1}$, $n_1=m$, and $n_2=n-m-1$, then we get

$$P_{WH(\Gamma(R,P))}(\lambda)=\lambda^{n-m-2}(\lambda+1)^{m-1}\left(\lambda^2-(m-1)\lambda-\frac{m(n-m-1)(n-2)^2}{(m-1)^2}\right).$$

□

Theorem 3.6. *The spectral radius of* $\Gamma(R,P)$ *associated with the Wiener matrix is*

$$\rho_{WH}(\Gamma(R,P))=\frac{m-1+\sqrt{(m-1)^2+\frac{4m(n-m-1)(n-2)^2}{(m-1)^2}}}{2}.$$

Proof. Based on Theorem 3.5, the roots of $P_{WH(\Gamma(R,P))}(\lambda)=0$ are eigenvalues of $WH(\Gamma(R,P))$. Therefore, we obtain $\lambda_1=-1$ with multiplicity $m-1$, $\lambda_2=0$ of multiplicity $n-m-2$, and $\lambda_{3,4}=\frac{m-1\pm\sqrt{(m-1)^2+\frac{4m(n-m-1)(n-2)^2}{(m-1)^2}}}{2}$. According to this fact, we get the spectrum of $\Gamma(R,P)$ as follows:

$$Spec_{WH}(\Gamma(R,P))=\begin{bmatrix}\lambda_3 & 0 & -1 & \lambda_4\\ 1 & n-m-2 & m-1 & 1\end{bmatrix}.$$

This leads to $\rho_{WH}(\Gamma(R,P))$ as the maximum absolute eigenvalue, and we complete the proof. □

In the following theorem, the Wiener-Hosoya energy of $\Gamma(R,P)$ is determined.

Theorem 3.7. *The Wiener-Hosoya energy of* $\Gamma(R,P)$ *is*

$$E_{WH}(\Gamma(R,P)) = m-1+\sqrt{(m-1)^2+\frac{4m(n-m-1)(n-2)^2}{(m-1)^2}}.$$

Proof. According to Theorem 3.6, the spectrum of $\Gamma(R,P)$ has been provided. Then,

$$\begin{aligned}E_{WH}(\Gamma(R,P)) =&(n-m-2)|0|+(m-1)|-1|+\left|\frac{m-1\pm\sqrt{(m-1)^2+\frac{4m(n-m-1)(n-2)^2}{(m-1)^2}}}{2}\right|\\ =&m-1+\sqrt{(m-1)^2+\frac{4m(n-m-1)(n-2)^2}{(m-1)^2}}.\end{aligned}$$

□

3.3 Distance signless Laplacian energy

This part focuses on the DSL-matrix of $\Gamma(R,P)$. Firstly, we present the characteristic polynomial of $\Gamma(R,P)$.

Theorem 3.8. *The characteristic polynomial of* $DSL(\Gamma(R,P))$ *is*

$$\begin{aligned}P_{DSL(\Gamma(R,P))}(\lambda) =&(\lambda-2n+m+5)^{n-m-2}(\lambda-n+3)^{m-1}\\ &\left(\lambda^2+(2m-5n+10)\lambda+4n^2-19n-2m^2+3m+21\right).\end{aligned}$$

Proof. Based on Theorem 3.1, the transmission matrix of $\Gamma(R,P)$, $T(\Gamma(R,P))$, is

$$\begin{array}{c} \\ p_1\\ p_2\\ \vdots\\ p_m\\ r_1\\ r_2\\ \vdots\\ r_{n-m-1}\end{array}\begin{array}{c}\begin{array}{cccccccc} p_1 & p_2 & \dots & p_m & r_1 & r_2 & \dots & r_{n-m-1}\end{array}\\ \begin{pmatrix} n-2 & 0 & \dots & 0 & 0 & 0 & \dots & 0\\ 0 & n-2 & \dots & 0 & 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & n-2 & 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0 & 2n-m-3 & 0 & \dots & 0\\ 0 & 0 & \dots & 0 & 0 & 2n-m-3 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 2n-m-3\end{pmatrix}\end{array}. \quad (2)$$

By Definition 2.3, and matrices in Equations (1) and (2), we obtain

$$DSL(\Gamma(R,P)) = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_m \\ r_1 \\ r_2 \\ \vdots \\ r_{n-m-1} \end{array} \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & \dots & p_m & r_1 & r_2 & \dots & r_{n-m-1} \end{array} \\ \begin{pmatrix} n-2 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & n-2 & \dots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & n-2 & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 2n-m-3 & 2 & \dots & 2 \\ 1 & 1 & \dots & 1 & 2 & 2n-m-3 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 2 & 2 & \dots & 2n-m-3 \end{pmatrix} \end{array}.$$

The above matrix can be partitioned into block matrices as follows:

$$DSL(\Gamma(R,P)) = \begin{pmatrix} (n-3)I_m + J_m & J_{m\times(n-m-1)} \\ J_{(n-m-1)\times m} & (2n-m-5)I_{n-m-1} + 2J_{n-m-1} \end{pmatrix}.$$

The characteristic polynomial of $DSL(\Gamma(R,P))$ is presented below:

$$P_{DSL(\Gamma(R,P))}(\lambda) = \begin{vmatrix} (\lambda-(n-3))I_m - J_m & -\frac{n-2}{m-1}J_{m\times(n-m-1)} \\ -\frac{n-2}{m-1}J_{(n-m-1)\times m} & (\lambda-(2n-m-5))I_{n-m-1} - 2J_{n-m-1} \end{vmatrix}. \quad (3)$$

We apply row and column operations to solve the above determinant. Let R_i be the i-th row and $R_i^{'}$ be the new i-th row obtained from row operation of $P_{DSL(\Gamma(R,P))}(\lambda)$. The same notations for column operation, we write as C_i and $C_i^{'}$. Then, by applying the following steps into Equation 3:

1. $R_{m+1+i} \longrightarrow R_{m+1+i} - R_{m+1}$, for $i = 1, 2, \dots, n-m-2$.
2. $R_{1+i} \longrightarrow R_{1+i} - R_1$, for $i = 1, 2, \dots, m-1$.
3. $C_{m+1} \longrightarrow C_{m+1} + C_{m+2} + \dots + C_{n-1}$.
4. $C_1 \longrightarrow C_1 + C_2 + \dots + C_m$.
5. $R_{m+1} \longrightarrow R_{m+1} - R_1$.
6. $C_1 \longrightarrow C_1 + \frac{\lambda-n+3}{\lambda-3n+2m+6}C_{m+1}$.

We obtain

$$P_{DSL(\Gamma(R,P))}(\lambda) = \begin{vmatrix} a & -J_{1\times(m-1)} & m+1-n & -J_{n-m-2} \\ 0_{(m-1)\times 1} & (\lambda-n+3)I_{m-1} & 0_{(m-1)\times 1} & 0_{m-1} \\ 0 & 0_{1\times(m-1)} & \lambda-3n+2m+6 & 0_{1\times(n-m-2)} \\ 0_{(n-m-2)\times 1} & 0_{(n-m-2)\times(m-1)} & 0_{(n-m-2)\times 1} & (\lambda-2n+m+5)I_{n-m-2} \end{vmatrix},$$

where $a = \frac{\lambda-n+3}{\lambda-3n+2m+6}(m+1-n) + \lambda - n - m + 3$. Therefore, we have

$$P_{DSL(\Gamma(R,P))}(\lambda) = (\lambda-2n+m+5)^{n-m-2}(\lambda-n+3)^{m-1} \left(\lambda^2 + (2m-5n+10)\lambda + 4n^2 - 19n - 2m^2 + 3m + 21\right).$$

□

Theorem 3.8 implies the following two results.

Theorem 3.9. *The spectral radius of* $\Gamma(R, P)$ *associated with the distance matrix is*

$$\rho_D(\Gamma(R,P)) = \frac{5n-2m-10+\sqrt{(2m-5n+10)^2-4(4n^2-19n-2m^2+3m+21)}}{2}.$$

Proof. According to Theorem 3.8, the roots of $P_{DSL(\Gamma(R,P))}(\lambda) = 0$ are eigenvalues of $DSL(\Gamma(R, P))$. Therefore, we obtain $\lambda_1 = n-3$ with multiplicity $m-1$, $\lambda_2 = 2n-m-5$ of multiplicity $n-m-2$, and $\lambda_{3,4} = \frac{5n-2m-10\pm\sqrt{(2m-5n+10)^2-4(4n^2-19n-2m^2+3m+21)}}{2}$. According to this fact, we get the spectrum of $\Gamma(R, P)$ as follows:

$$Spec_{DSL}(\Gamma(R,P)) = \begin{bmatrix} \lambda_3 & 2n-m-5 & n-3 & \lambda_4 \\ 1 & n-m-2 & m-1 & 1 \end{bmatrix}.$$

As a result, we obtain the maximum absolute eigenvalue to be the spectral radius of $\Gamma(R, P)$, and the proof is completed. □

Theorem 3.10. *The distance signless Laplacian energy of* $\Gamma(R, P)$ *is*

$$E_{DSL}(\Gamma(R,P)) = 2n^2 + m^2 - 2mn - 5n + 2m + 3.$$

Proof. According to the spectrum of $\Gamma(R, P)$ in the proofing part of Theorem 3.9, DSL-energy of $\Gamma(R, P)$ is given by

$$\begin{aligned} E_{DSL}(\Gamma(R,P)) =& (n-m-2)|2n-m-5| + (m-1)|n-3| + \\ & \left|\frac{5n-2m-10\pm\sqrt{(2m-5n+10)^2-4(4n^2-19n-2m^2+3m+21)}}{2}\right| \\ =& (n-m-2)(2n-m-5) + (m-1)(n-3) + 5n-2m-10 \\ =& 2n^2 + m^2 - 2mn - 5n + 2m + 3. \end{aligned}$$

□

4 Discussion

The distance energy formula has R^2 value of 0.914, while R^2 of the Wiener-Hosoya energy is 0.828, and the distance signless Laplacian energy has $R^2 = 0.955$. It is presented in Figures 1, 2, and 3. Overall, having R^2 values close to 1 indicates that the formula is highly effective in explaining the variation in the data. It suggests a strong fit between the model and the observed data, indicating that the formula likely captures meaningful relationships between the variables. However, as always, it's important to consider other aspects of model evaluation and potential limitations of the analysis.

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Change Statistics				
					R Square Change	F Change	df1	df2	Sig. F Change
1	.956[a]	.914	.909	12.46969	.914	175.987	2	33	.000

a. Predictors: (Constant), m, n

Figure 1: The Distance Energy of $\Gamma(R, P)$.

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Change Statistics				
					R Square Change	F Change	df1	df2	Sig. F Change
1	.910[a]	.828	.818	5.26302	.828	79.446	2	33	.000

a. Predictors: (Constant), m, n

Figure 2: The Wiener-Hosoya Energy of $\Gamma(R, P)$.

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Change Statistics				
					R Square Change	F Change	df1	df2	Sig. F Change
1	.977[a]	.955	.952	6.693	.955	350.063	2	33	.000

a. Predictors: (Constant), m, n

Figure 3: The Distance Signless Laplacian Energy of $\Gamma(R, P)$.

Moreover, from all above, the energy and spectral radius results show the relationships between both values as presented below:

Corollary 4.1. *In* $\Gamma(R, P)$,

1. $E_D(\Gamma(R, P)) = 2 \cdot \rho_D(\Gamma(R, P))$.
2. $E_{WH}(\Gamma(R, P)) = 2 \cdot \rho_{WH}(\Gamma(R, P))$.

Corollary 4.2. *In* $\Gamma(R, P)$,

$$E_D(\Gamma(R, P)) < E_{WH}(\Gamma(R, P)) < E_{DSL}(\Gamma(R, P)).$$

5 Conclusion

From the earlier discussion, we have presented the energy formulas prime ideal graph of a commutative ring based on transmission-based matrices. The distance signless Laplacian energy is the highest and the distance energy is the lowest. Additionally, the energy is twice the spectral radius associated with distance and Wiener-Hosoya matrices.

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Conflicts of Interest The authors declare no conflict of interest.

References

[1] D. F. Anderson, T. Asir, A. Badawi & T. T. Chelvam (2021). *Graphs from Rings*. Springer Nature, Switzerland. https://link.springer.com/book/10.1007/978-3-030-88410-9.

[2] P. M. Cohn (2012). *Basic algebra: Groups, rings and fields*. Springer Science & Business Media, London.

[3] I. Gutman (1978). The energy of graph. *Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz*, *103*, 1–22.

[4] R. A. Horn & C. R. Johnson (2012). *Matrix analysis*. Cambridge University Press, Cambridge, United Kingdom. https://doi.org/10.1017/CBO9780511810817.

[5] H. Ibrahim, R. Sharafdini, T. Réti & A. Akwu (2021). Wiener-Hosoya matrix of connected graphs. *Mathematics*, *9*(4), 359. https://doi.org/10.3390/math9040359.

[6] G. Indulal, I. Gutman & A. Vijayakumar (2008). On distance energy of graphs. *MATCH Communications in Mathematical and in Computer Chemistry*, *60*(2), 461–472.

[7] M. A. Khan (2021). Contributions to generalized derivation on prime near-ring with its application in the prime graph. *Malaysian Journal of Mathematical Sciences*, *15*(1), 109–123.

[8] S. Malekpour & B. Bazigaran (2020). Some results on the graph associated to a lattice with given a filter. *Malaysian Journal of Mathematical Sciences*, *14*(3), 533–541.

[9] S. Pirzada & M. A. U. Haq (2023). On the spread of the distance signless Laplacian matrix of a graph. *Acta Universitatis Sapientiae, Informatica*, *15*(1), 38–45. https://doi.org/10.2478/ausi-2023-0004.

[10] H. S. Ramane & S. S. Shinde (2017). Degree exponent polynomial of graphs obtained by some graph operations. *Electronic Notes in Discrete Mathematics*, *63*, 161–168. https://doi.org/10.1016/j.endm.2017.11.010.

[11] N. Rehman, Nazim & M. Nazim (2024). Exploring normalized distance Laplacian eigenvalues of the zero-divisor graph of ring $\mathbb{Z}_n$. *Rendiconti del Circolo Matematico di Palermo Series 2*, *73*(2), 515–526. https://doi.org/10.1007/s12215-023-00927-y.

[12] M. U. Romdhini, A. Nawawi & C. Y. Chen (2023). Neighbors degree sum energy of commuting and non-commuting graphs for dihedral groups. *Malaysian Journal of Mathematical Sciences*, *17*(1), 53–65. https://doi.org/10.47836/mjms.17.1.05.

[13] M. U. Romdhini & A. Nawawi (2024). Degree sum exponent distance energy of non-commuting graph for dihedral groups. *Italian Journal of Pure and Applied Mathematics*, *51*, 427–442.

[14] M. U. Romdhini, A. Nawawi, F. Al-Sharqi & A. Al-Quran (2024). Spectral properties of power graph of dihedral groups. *European Journal of Pure and Applied Mathematics*, *17*(2), 591–603. https://doi.org/10.29020/nybg.ejpam.v17i2.5036.

[15] M. U. Romdhini, A. Nawawi, F. Al-Sharqi, A. Al-Quran & S. R. Kamali (2024). Wiener-Hosoya energy of non-commuting graph for dihedral groups. *Asia Pacific Journal of Mathematics*, *11*(9), 1–9. https://doi.org/10.28924/APJM/11-9.

[16] H. M. Salih & A. A. Jund (2022). Prime ideal graphs of commutative rings. *Indonesian Journal of Combinatorics*, *6*(1), 42–49. http://dx.doi.org/10.19184/ijc.2022.6.1.2.

[17] A. G. Syarifudin, I. Muchtadi-Alamsyah & E. Suwastika (2024). Topological indices and properties of the prime ideal graph of a commutative ring and its line graph. *Contemporary Mathematics*, *5*(2), 1342–1354. https://doi.org/10.37256/cm.5220243574.

[18] N. A. F. O. Zai, N. H. Sarmin, S. M. S. Khasraw, I. Gambo & N. Zaid (2023). On the non-zero divisor graphs of some finite commutative rings. *Malaysian Journal of Mathematical Sciences*, *17*(2), 105–112. https://doi.org/10.47836/mjms.17.2.02.